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A NOTE ON THE "ORDER OF APPROXIMATION"

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INTRODUCTION

In finite difference approximations the "order of approximation" is generally obtained by considering the errors in representation of the differential equation at a mesh point by the values of the unknown function at the surrounding mesh points. Thus, for instance, if a simple one-dimensional approximation to a second-order equation

$$\frac{d^2\phi}{dx^2} + f(x) = 0 \tag{1a}$$

is made, we have, by Taylor's expansion around point 'i', on a mesh of size h

$$\phi_{i-1} - 2\phi_i + \phi_i + h^2 f(x_i) + O(h^4) = 0. \tag{1b}$$

Here the approximation is said to be *fourth-order accurate*[1]. This clearly applies to the error in ϕ at the mesh points.

In the finite element method (or indeed in other trial function approximations) where the function ϕ is approximated as

$$\phi \approx \hat{\phi} = \sum_i N_i \phi_i, \tag{2}$$

and the approximating equations are derived by a weighting (or Galerkin) method as integral statements such as

$$\int_\Omega W_j \left(\frac{d^2\hat{\phi}}{dx^2} + f \right) dx = 0 \tag{3}$$

a different viewpoint is taken. Here the error is generally considered *at all points in the domain* rather than at nodes alone. Thus, for instance, if, in the example cited, locally linear trial functions are taken it is evident that errors are of the order $O(h^2)$ and that the approximation would be deemed to be only *second order accurate*. This is confusing as it is well known that both procedures will in the present example of a regular mesh yield identical discrete equations and hence nodal results (at least for the case of constant $f(x)$).

The differences lie obviously in the definitions only; the first one being concerned with accuracy of nodes only—the latter with global accuracy. Thus if for instance we wish to consider approximation in the energy or other global norms only the finite element definition is capable of supplying the correct answer but for nodal values the finite difference approach suffices—and indeed for such points superconvergence may occur.

A theorem due to Pin Tong[2] demonstrates that, for one dimensional problems which are self adjoint and in which the trial functions include solution of the homo-

geneous differential equation, exact nodal values will be found for all $f(x)$ and thus in local (finite difference) terms the order of error is $O(h^*)$. The theorem may be extended to problems which are not self adjoint by employing Green's theorem and using weighting functions which are exact solutions to the homogeneous adjoint differential equation. In addition, the theorems provide a method to compute exact values of the derivatives of ϕ at the interelement nodes.

In what follows we shall show a simple procedure for finding the 'nodal' order of error (and hence order of accuracy) for some finite element models, in particular demonstrating the superconvergence of one-dimensional problems of the Pin Tong kind. Further, we shall comment on orders of error associated with some time stepping algorithms derived by the 'finite element' method. We trust that the reader will be convinced that the simple statements often made on order of method in the latter context should be correctly investigated.

NODAL VERSUS LOCAL ERROR IN ELLIPTIC PROBLEMS

Consider a slightly more general one-dimensional equation than eqn (1a) in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) + q\phi + f = 0, \tag{4}$$

and let this be approximated in the usual finite element manner (eqns 2–3) using the Galerkin method and linear element functions (Fig. 1). Immediately, the approximating equation is obtained as

$$K_{ij}\phi_j = \gamma_i \tag{5}$$

where the element contributions are

$$K_{ij}^e = \int_{\Omega^e} \left| -\frac{\mathrm{d}N_i}{\mathrm{d}x} k \frac{\mathrm{d}N_i}{\mathrm{d}x} + qN_iN_j \right| \mathrm{d}x; \quad \gamma_i^e = -\int_{\Omega^e} N_i f \, \mathrm{d}x, \tag{6}$$

and ϕ_j lists the nodal values of ϕ.

The approximation to ϕ locally has an error of $O(h^2)$ and the global energy of $O(h^2)$, as is well known[3, 4].

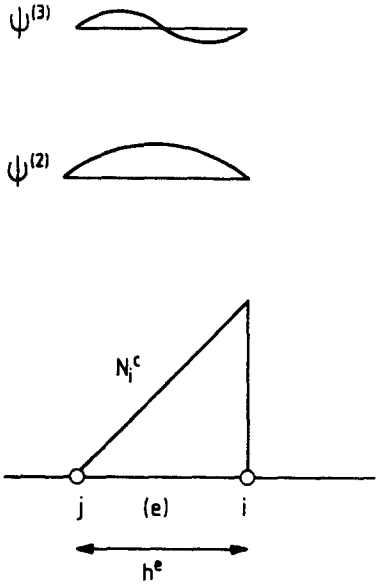

Fig. 1. A one dimension element. Linear and higher order hierarchical functions.

Consider, however, the changes which would be introduced by augmenting the approximation of eqn 2 by adding in each element hierarchical polynomials of order p, i.e. writing

$$\hat{\phi} = \sum N_i \phi_i + \alpha_p \psi_p, \tag{7}$$

these polynomials being so chosen as to have zero nodal values (Fig. 1).

Now the typical equation for node i is augmented by element contributions

$$K_{ip}^e \alpha_p = \int_{\Omega^e} \left(- \frac{dN_i}{dx} k \frac{d\psi_p}{dx} dx + q N_i \psi_p \right) dx \alpha_p. \tag{8}$$

In general, these terms are not zero, but if we take k constant in each element and q zero (for which the Pin Tong criterion now holds), this contribution disappears as

$$\int_{\Omega^e} \frac{dN_i}{dx} k \frac{d\psi_p}{dx} dx = \frac{k}{h^e} \int_{\Omega^e} \frac{d\psi_p}{dx} dx = \frac{k}{h^e} \psi_p \Big|_i^j \equiv 0 \tag{9}$$

and we have defined the function ψ_p with zero values at ends of each element.

Clearly the equations for determining the nodal variables ϕ are not affected by additional polynomials $p = 2, 3, \ldots, \infty$ and hence nodal error is of $O(h^\infty)$ and results are superconvergent at the interelement nodes. The approximations to ϕ are, however, affected by the ψ_p functions within each element and improved accuracy can be achieved at all points of the domain and the energy by their use.

We leave the proof of superconvergence of fourth order (beam) equations to the reader who can also demonstrate that the superconvergence property applies only to end points of higher order elements (as α_p is not zero in above arguments).

TIME-STEPPING ALGORITHM

Consider the first-order equation

$$C\dot{\phi} + K\phi + f = 0 \tag{10}$$

representing a typical problem of say transient heat conduction for which a time-stepping algorithm is derived by the 'finite element process.' Assuming, for instance, a linear variation of the function $\phi(t)$ in a time interval, we can write

$$\begin{aligned} \phi(t) &= \phi^n + \frac{t}{\Delta t}(\phi^{n+1} - \phi^n) \\ &= \phi^n + \frac{t}{\Delta t}(\Delta\phi^n), \end{aligned} \tag{11}$$

where t is the difference in time n and ϕ^n denoted the approximate value to ϕ at n.

Inserting eqn (11) into (10) and writing the weighted equation results in the well-known θ algorithm[4]

$$C\Delta\phi^n + \Delta t K(\phi^n + \theta\Delta\phi^n) + \Delta t \bar{f} = 0, \tag{12}$$

where

$$\theta = \frac{\int_0^{\Delta t} Wt \, dt}{\Delta t \int_0^{\Delta t} W \, dt} \quad \text{and} \quad \bar{f} = \frac{\int_0^{\Delta t} Wf \, dt}{\int_0^{\Delta t} W \, dt}. \tag{13}$$

It is evident that in the finite element sense the approximation has an error of $O(\Delta t^2)$ in each time step as an exact linear solution is exactly reproduced. If 'nodal' values (ϕ'', etc.) are considered, the same accuracy applies unless we can show some superconvergence property by the arguments of the preceding section. Thus, let us *consider the augmentation of expression (11) by a quadratic function* $\psi_2 = t(\Delta t - t)$ etc. in the hierarchical manner, i.e.

$$\phi(t) = \phi'' + \frac{t}{\Delta t}\Delta\phi'' + \alpha_2\psi_2. \tag{14}$$

The 'weighted residual' eqn (11) is augmented by

$$\frac{\Delta t\left[\int_0^{\Delta t} W(C\dot{\psi}_2 + K\psi_2)\,dt\right]}{\int_0^{\Delta t} W\,dt} \equiv C(1 - 2\theta) + \Delta t K(\theta - \beta), \tag{15}$$

where

$$\beta = \frac{\int_0^{\Delta t} Wt^2\,dt}{\Delta t^2\int_0^{\Delta t} W\,dt}. \tag{16}$$

This addition is generally non-zero, indicating that, at nodes (or time stations), errors of $O(\Delta t^2)$ only holds. However, for

$$\theta = 1/2 \quad \text{and} \quad \beta = \theta \tag{17}$$

quadratic terms do not affect the solution and an error of $O(\Delta t^3)$ is achieved. This is well known to be the accuracy of the $\theta = 1/2$ or Crank-Nicolson method in finite difference estimates (which is called a second-order accurate method). Here, however, the analysis reveals that this is true only if the weighting function W satisfy an auxilliary condition and $\bar{f}$ is accordingly calculated.

The procedure can be applied quite generally to all finite element derived time marching schemes. Consider for instance a *single step* scheme starting from known values of ϕ'' and $\dot{\phi}''$ in which a second-order approximation is made to $\phi(t)$[5], i.e.

$$\phi = \phi'' + \dot{\phi}''t + \mathrm{d}''t^2, \tag{18}$$

where we note that d'' may be expressed alternatively in terms of ϕ^{n+1} as

$$\mathrm{d}'' = (\phi^{n+1} - \phi'' - \dot{\phi}''\,\Delta t)/\Delta t^2. \tag{19}$$

The weighted approximation to eqn (10) yields

$$C(\dot{\phi}'' + \theta\,\Delta t\,\mathrm{d}'') + K(\phi'' + \theta\,\Delta t\dot{\phi}'' + \beta\,\Delta t^2\,\mathrm{d}'') + \Delta t\bar{f} = 0, \tag{20}$$

where β is defined in eqn (16), and from which d'' is found. This time-step algorithm gives an order of error $O(\Delta t^3)$ at all points of the range.

To consider only the error in values at n or $n + 1$, we investigate the addition to eqn (18) of

$$\alpha_3\psi_3 = (t^3 - \Delta t t^2)\alpha_3, \tag{21}$$

i.e. a hierarchical cubic to the function.

Proceeding as before, we obtain for the nodal approximation to eqn (20) an augmented term due to $\alpha_3\psi_3$

$$\frac{\int_0^{\Delta t} W[C\dot{\psi}_3 + K\psi_3]\,dt}{\Delta t^2 \int_0^{\Delta t} W\,dt} \equiv C(3\beta - 2\theta) + \Delta t K(\beta^* - \beta), \tag{22}$$

where

$$\beta^* = \int_0^{\Delta t} Wt^3\,dt/\Delta t^3 \int_0^{\Delta t} W\,dt. \tag{23}$$

Once again one higher order error is found (independently of C and K) for

$$\beta = \tfrac{2}{3}\theta \tag{24}$$

and the appropriate weighting function has to satisfy the constraint, $\beta^* = \beta$. These values must also be employed when considering the calculation of $\bar{f}$. In addition, stability must be investigated to ensure that the algorithm is convergent.

The procedure outlined is obvious and general and allows a simple determination of time marching parameters resulting in higher order approximations.

This is quite useful in evaluating further some of the time-marching schemes.

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